

Growth, Oscillation, and the Death Spiral:

Closed-form parameter estimates for common mathematical models

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Abstract. Closed-form parameter estimates are given for models of exponential growth, logistic growth, the Gompertz curve, Korf growth, simple harmonic motion, damped vibrations, and the logarithmic spiral. The study employs the method of *Regressions et Equations Integrales* (Jean Jacquelin, 2014), but presents more concise expressions in some instances, and gives solutions to some problems not considered in that work. Computer programs implementing the method are described.

Introduction

Parameter estimation for nonlinear models remains a major topic in applied mathematics, with applications spanning fields from biology to physics and engineering. While numerical optimization techniques are widely employed, closed-form solutions offer distinct advantages: they provide analytical insight into parameter dependencies, eliminate convergence concerns, and enable rapid computation without iterative procedures.

The integral method, pioneered by Jean Jacquelin, represents an elegant approach to the problem. By transforming nonlinear regression into a system of linear equations through strategic integration, this technique circumvents the need for iterative optimization. The method is particularly advantageous for models where iterative least-squares approaches struggle with local minima or require careful initialization.

This article extends Jacquelin's method to derive new closed-form parameter estimates for important classes of models. The Mitscherlich equation, known in population biology as the monomolecular model, is also widely used in chemical kinetics to describe processes approaching an asymptotic limit. Next covered are oscillatory systems, which model phenomena ranging from mechanical vibrations to electrical circuits. Finally is addressed the logarithmic (equiangular) spiral with arbitrary center coordinates—a curve fundamental to

modeling growth patterns in nature, from nautilus shells to galactic arms. The inclusion of translational parameters significantly complicates the estimation problem compared to the centered case.

For each model class, explicit parameter estimators are derived. It is shown that the closed-form solutions provide excellent initial estimates for refinement procedures. The resulting formulas are computationally efficient and numerically stable, making them suitable for real-time applications and large-scale data analysis.

Background

Roots of the present method extend deep to the work of Hartley (1948), who presented a method for parameter estimation of growth models that exploited finite differences. He showed that through simple substitutions, a method obtained for the Mitscherlich equation could be extended to apply to the Gompertz and Verhulst equations as well.

Although similar in spirit, the method of Jacquelin (2014) differs in details. His small book, *Regressions et Equations Integrales*, developed a general and powerful method that converts many nonlinear models into linear systems of equations in terms of integrals. These may be satisfactorily approximated by the trapezoid rule, and thus are known quantities. The present article owes much to this prior work.

The derivation which follows demands no mathematics beyond ordinary calculus. The resulting formulas and programs may be applied without understanding of the derivations.

Original contributions

- Simplified formulas for diminishing returns and sinusoidal models
- Treatment of Korf growth
- Treatment of damped harmonic motion about an unknown center
- Treatment of the logarithmic spiral about an unknown center

Derivation

Growth

Exponential Growth. Begin with the simple model $z = e^{rx}$ and integrate. For reasons that will be explained in the next section, the constant of integration may be neglected. Letting s denote the integral of z with respect to x ,

$$s = \int z \, dx = \frac{e^{rx}}{r} = \frac{z}{r}$$
$$z = rs$$

To minimize the squared-error ϵ between the model predictions z_i and the observations y_i over a data set of size N , write

$$\epsilon = \frac{1}{2} \sum_{i=1}^N (z_i - y_i)^2 = \frac{1}{2} \sum_{i=1}^N (r s_i - y_i)^2$$

dropping subscripts for concision,

$$\begin{aligned} \frac{\partial \epsilon}{\partial r} &= \sum (r s - y) s = 0 \\ r \sum s^2 - \sum s y &= 0 \end{aligned}$$

$$r = \frac{\sum \tilde{s} y}{\sum \tilde{s}^2}$$

which will hold when $\tilde{s}$ is a linearly detrended function $\tilde{s} = s - \hat{s}_{lin}$, by subtracting the least squares line of s as a function of x .

Law of diminishing returns. The Mitscherlich model of plant growth posits that each additional unit of nutrient will produce a smaller increment of growth than the proceeding unit. (Fox, 1971) This is expressed by the equation

$$\frac{dy}{dx} = c(a - y)$$

where x is the growth factor, y is the yield, a is the maximum attainable yield, and c is a constant. Integrating gives

$$\begin{aligned} \int \frac{1}{a-y} dy &= \int c dx \\ -\ln(a-y) &= cx + k_1 \end{aligned}$$

Rename the exponentiated constant of integration $b = \exp(k_1)$, then

$$y - a = -b e^{-cx}$$

The diminishing returns model with z as the prediction is thus:

$$z = a - b e^{-cx}$$

Let s denote the integral of z with respect to x . Then,

$$s = \int z dx = \int (a - b e^{-cx}) dx = ax + \frac{b}{c} e^{-cx} + k_2$$

$$z + cs = a + acx + ck_2$$

$$z = -cs + acx + a + ck_2$$

Following the method of Jacquelin (2014), let

$$A_0 = -c$$

$$B_0 = ac$$

$$C_0 = a + ck_2$$

The error residual over a set of N data of predicted z_i and observed y_i may be written

$$\epsilon = \frac{1}{2} \sum_{i=1}^N (A_0 s_i + B_0 x_i + C_0 - y_i)^2$$

Setting $\frac{\partial \epsilon}{\partial A_0} = \frac{\partial \epsilon}{\partial B_0} = \frac{\partial \epsilon}{\partial C_0} = 0$ yields the system of linear equations

$$\begin{bmatrix} \sum s^2 & \sum xs & \sum s \\ \sum xs & \sum x^2 & \sum x \\ \sum s & \sum x & N \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \\ C_0 \end{bmatrix} = \begin{bmatrix} \sum sy \\ \sum xy \\ \sum y \end{bmatrix}$$

An integral shifted by a linear function is equivalent to the integral of the original function shifted by a constant. Since the model contains an additive constant as an unknown parameter, it is permissible to detrend s as a function of x , $\tilde{s} = s - \hat{s}_{lin}$. Observe that

$$\begin{aligned} \sum \tilde{s} &= 0 \\ \sum x \tilde{s} &= 0 \end{aligned}$$

Apply Cramer's rule to solve the system of equations. Developing the determinant of the matrix along the first row, only the leftmost element is nonzero,

$$A_0 = \frac{\sum \tilde{s} y}{\sum \tilde{s}^2}$$

$$c = -\frac{\sum \tilde{s} y}{\sum \tilde{s}^2}$$

Since the exponential growth model is a special case of the diminishing returns model, it can be seen that it was permissible to detrend that integral likewise. With parameter c determined, the problem of determining a and b reduces to linear least squares.

Gompertz curve. The Gompertz model was introduced as an actuarial model for predicting human lifetimes, but has also found use as a growth model. It has the equation

$$z = a e^{-b e^{-cx}}$$

where population z is a function of time x , and a , b and c are parameters to be determined. Linearize based on the method of Hartley (1948)

$$\ln(z) = \ln(a) - b e^{-cx}$$

Making the substitutions

$$\begin{aligned} z' &= \ln(z) \\ a' &= \ln(a) \end{aligned}$$

the problem is in the form of the diminishing returns model, and may be solved by the prior method.

Logistic growth. The well-known logistic growth model has the equation

$$z = \frac{K}{1 + b e^{-rx}}$$

where population z is a function of time x , r is an intrinsic rate of growth, K is the carrying capacity of the environment, and b is a parameter which depends on the initial conditions. Making the inverse transformation (Hartley, 1948)

$$\frac{1}{z} = \frac{1}{K} + \frac{b}{K} e^{-rx}$$

Make the substitutions

$$\begin{aligned}
z' &= \frac{1}{z} \\
a' &= \frac{1}{K} \\
b' &= \frac{b}{K} \\
c' &= r
\end{aligned}$$

The problem is now in the form of the diminishing returns model and may be solved by that method.

Korf growth. The Chapman-Richards four parameter growth model has been very widely used. It is unfortunately quite resistant to integration or any closed-form estimate of its parameters. As a consolation, one additional growth model shall be considered here out the many which have been proposed in the literature. The Korf equation has been found accurate for modeling the growth of tree trunk diameter. (Zeide, 1993) It has the form

$$z = a e^{-bx^{-c}}$$

Take the logarithm:

$$\ln(z) = \ln(a) - bx^{-c}$$

Integrate:

$$\begin{aligned}
s &= \int \ln(z) dx = x \ln(a) - \frac{b}{-c+1} x^{-c+1} + k \\
\frac{1-c}{x} s &= (1-c) \ln(a) - bx^{-c} + \frac{1-c}{x} k
\end{aligned}$$

Combining with the above yields

$$\ln(z) = (1-c) \frac{s}{x} + (c-1) k \frac{1}{x} + c \ln(a)$$

Define:

$$\begin{aligned}
A_0 &= 1-c \\
B_0 &= (c-1)k \\
C_0 &= c \ln(a)
\end{aligned}$$

Applying Jacquelin's method, write the error of approximation as

$$\epsilon = \frac{1}{2} \sum (\ln(z) - \ln(y))^2 = \sum (A_0 \frac{s}{x} + B_0 \frac{1}{x} + C_0 - \ln(y))^2$$

Let $\frac{\partial \epsilon}{\partial A_0} = \frac{\partial \epsilon}{\partial B_0} = \frac{\partial \epsilon}{\partial C_0} = 0$, then the system of linear equations is

$$\begin{bmatrix} \sum \frac{s^2}{x^2} & \sum \frac{s}{x^2} & \sum \frac{s}{x} \\ \sum \frac{s}{x^2} & \sum \frac{1}{x^2} & \sum \frac{1}{x} \\ \sum \frac{s}{x} & \sum \frac{1}{x} & N \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \\ C_0 \end{bmatrix} = \begin{bmatrix} \sum \frac{s}{x} \ln(y) \\ \sum \frac{1}{x} \ln(y) \\ \sum \ln(y) \end{bmatrix}$$

The growth rate is given by $c = 1 - A_0$, and a and b may be determined by linear regression.

Oscillation

Simple harmonic motion. Simple harmonic motion arises in a variety of situations, and numerous techniques have been developed to determine the frequency. The integral method presented here requires only modest computation, and does not require the observations to be equally spaced. Begin by integrating the model

$$\begin{aligned} z &= a \cos(\omega x) + b \sin(\omega x) + c \\ s_1 &= \int z \, dx = \frac{a}{\omega} \sin(\omega x) - \frac{b}{\omega} \cos(\omega x) + cx + k_1 \\ s_2 &= \int s_1 \, dx = -\frac{a}{\omega^2} \cos(\omega x) - \frac{b}{\omega^2} \sin(\omega x) + \frac{c}{2} x^2 + k_1 x + k_2 \\ s_2 &= \frac{-1}{\omega^2} z + \frac{c}{2} x^2 + k_1 x + k_2 \end{aligned}$$

Thus,

$$z = -\omega^2 s_2 + \frac{c \omega^2}{2} x^2 + k_1 \omega^2 x + k_2 \omega^2$$

Following Jacquelin, define

$$A_0 = -\omega^2$$

$$B_0 = \frac{c \omega^2}{2}$$

$$C_0 = k_1 \omega^2$$

$$D_0 = k_2 \omega^2$$

The loss function of squared differences between the observations and predictions is

$$\epsilon = \frac{1}{2} \sum (z - y)^2 = (A_0 s_2 + B_0 x^2 + C_0 x + D_0 - y)^2$$

Setting $\frac{\partial \epsilon}{\partial A_0} = \frac{\partial \epsilon}{\partial B_0} = \frac{\partial \epsilon}{\partial C_0} = \frac{\partial \epsilon}{\partial D_0} = 0$ gives the system of linear equations

$$\begin{bmatrix} \sum s_2^2 & \sum x^2 s_2 & \sum x s_2 & \sum s_2 \\ \sum x^2 s_2 & \sum x^4 & \sum x^3 & \sum x^2 \\ \sum x s_2 & \sum x^3 & \sum x^2 & \sum x \\ \sum s_2 & \sum x^2 & \sum x & N \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \\ C_0 \\ D_0 \end{bmatrix} = \begin{bmatrix} \sum s_2 y \\ \sum x^2 y \\ \sum x y \\ \sum y \end{bmatrix}$$

The second integral shifted by a parabola is equivalent to the second integral of the original function shifted by a constant. Since the model contains an additive constant as an unknown parameter, it is permissible to parabolically detrend the second integral as $\tilde{s}_2 = s_2 - \hat{s}_{2prb}$ by subtracting the least-squares parabola of s_2 as a function of x . Then notice that

$$\begin{aligned} \sum \tilde{s}_2 &= 0 \\ \sum x \tilde{s}_2 &= 0 \\ \sum x^2 \tilde{s}_2 &= 0 \end{aligned}$$

Apply Cramer's rule to solve the system. Developing the determinant along the first row, only the upper-left element is nonzero, and so

$$A_0 = \frac{\sum \tilde{s}_2 y}{\sum \tilde{s}_2^2}$$

$$\omega = \sqrt{-\frac{\sum \tilde{s}_2 y}{\sum \tilde{s}_2^2}}$$

Now that frequency has been determined, parameters a , b and c may be determined by linear least-squares.

Damped vibrations. Consider a mass moving in damped harmonic motion. The case in which the rest position is at zero has been treated by Jacquelin. Consider now a more general case. Suppose the rest position is at an unknown offset k_0 , and

$$z = e^{rx} (a \cos(\omega x) + b \sin(\omega x)) + k_0 = c e^{rx} \cos(\omega x - \phi) + k_0$$

where z is position as a function of time x , r is the decay rate, ω is the angular frequency, and parameters a and b are related to the amplitude c and phase angle ϕ by

$$c = \sqrt{a^2 + b^2}$$

$$\phi = \arctan(b, a)$$

$$a = c \cos(\phi)$$

$$b = c \sin(\phi)$$

The integral equation, by aid of computer algebra (Maxima), is

$$z = -us_2 + 2rs_1 + \frac{k_0 u}{2} x^2 + (k_1 u - 2k_0 r)x + k_2 u - 2k_1 r + k_0$$

where $u = r^2 + \omega^2$

Let

$$A_0 = -u$$

$$B_0 = 2r$$

$$C_0 = \frac{k_0 u}{2}$$

$$D_0 = (k_1 u - 2k_0 r)$$

$$E_0 = k_2 u - 2k_1 r + k_0$$

The loss function of sum of squared differences between the predictions and observations is

$$\epsilon = \frac{1}{2} \sum (A_0 s_2 + B_0 s_1 + C_0 x^2 + D_0 x + E_0 - y)^2$$

Setting $\frac{\partial \epsilon}{\partial A_0} = \frac{\partial \epsilon}{\partial B_0} = \frac{\partial \epsilon}{\partial C_0} = \frac{\partial \epsilon}{\partial D_0} = \frac{\partial \epsilon}{\partial E_0} = 0$ yields the system of linear equations:

$$\begin{bmatrix} \sum s_2^2 & \sum s_1 s_2 & \sum x^2 s_2 & \sum x s_2 & \sum s_2 \\ \sum s_1 s_2 & \sum s_1^2 & \sum x^2 s_1 & \sum x s_1 & \sum s_1 \\ \sum x^2 s_2 & \sum x^2 s_1 & \sum x^4 & \sum x^3 & \sum x^2 \\ \sum x s_2 & \sum x s_1 & \sum x^3 & \sum x^2 & \sum x \\ \sum s_2 & \sum s_1 & \sum x^2 & \sum x & N \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \\ C_0 \\ D_0 \\ E_0 \end{bmatrix} = \begin{bmatrix} \sum s_2 y \\ \sum s_1 y \\ \sum x^2 y \\ \sum x y \\ \sum y \end{bmatrix}$$

The fortunate cancellations in the case of undamped motion do not occur here, and it will be necessary to solve the system. The decay rate and angular frequency parameters are

$$r = \frac{B_0}{2}$$

$$\omega = \sqrt{-(A_0 + r^2)}$$

Parameters a , b , and k_0 may now be determined by linear least-squares. Jacquelin calls the forgoing procedure the “short way” and follows with a complex method for re-estimating the phase and frequency as a linear regression of transformed variables. This adds considerable difficulty, and the interested reader is referred to pages 64-70 of *Regressions et Equations Integrales* for the “full way.”

The death spiral

Consider the case of a logarithmic spiral. If the spiral is centered on the origin, its equation may be written in polar form $r = a e^{b\theta}$. Using logarithms, this may be written

$$\ln(r) = \ln(a) + b\theta$$

and the parameters $\ln(a)$ and b may be trivially recovered by linear least-squares. The following method extends to the case of spirals which are not centered on the origin. A logarithmic spiral may be written in complex form as

$$z = c \exp((r + i\omega)x - i\phi)$$

where z is the complex position of real time x , and c is the real amplitude, r is the real decay rate, ω is the real angular frequency, and ϕ is the real phase angle, and i is the imaginary unit. Neglecting the constant of integration, the complex integral s is

$$s = \int z dx = \frac{c \exp((r + i\omega)x - i\phi)}{r + i\omega} = \frac{z}{r + i\omega}$$

$$z = (r + i\omega) s$$

Separating the real and complex parts of z gives

$$\Re(z) = r \Re(s) - \omega \Im(s)$$

$$\Im(z) = \omega \Re(s) + r \Im(s)$$

To optimize on the complex plane, recall that the squared distance metric is

$$\|z - y\|^2 = \Re(z - y)^2 + \Im(z - y)^2$$

Thus,

$$\epsilon = \frac{1}{2} \sum (r \Re(s) - \omega \Im(s) - \Re(y))^2 + (\omega \Re(s) + r \Im(s) - \Im(y))^2$$

Setting $\frac{\partial \epsilon}{\partial r} = 0$ yields

$$r = \frac{\sum \Re(s) \Re(y) + \sum \Im(s) \Im(y)}{\sum \Re(s)^2 + \sum \Im(s)^2}$$

Setting $\frac{\partial \epsilon}{\partial \omega} = 0$ yields

$$\omega = \frac{\sum \Re(s) \Im(y) - \sum \Im(s) \Re(y)}{\sum \Re(s)^2 + \sum \Im(s)^2}$$

To find phase and amplitude, recall the subtraction formulas:

$$\cos(\omega x - \phi) = \cos(\omega x) \cos(\phi) + \sin(\omega x) \sin(\phi)$$

$$\sin(\omega x - \phi) = \sin(\omega x) \cos(\phi) - \cos(\omega x) \sin(\phi)$$

and Euler's formula, $e^{i\phi} = \cos(\phi) + i \sin(\phi)$

The spiral equation can then be expressed as

$$\Re(z) = c e^{rx} (\cos(\omega x) \cos(\phi) + \sin(\omega x) \sin(\phi))$$

$$\Im(z) = c e^{rx} (\sin(\omega x) \cos(\phi) - \cos(\omega x) \sin(\phi))$$

Let $a = c \cos(\phi)$ and $b = c \sin(\phi)$, then

$$\Re(z) = a e^{rx} \cos(\omega x) + b e^{rx} \sin(\omega x)$$

$$\Im(z) = a e^{rx} \sin(\omega x) - b e^{rx} \cos(\omega x)$$

This is a linear system of equations for the unknown parameters a and b which may be solved by least-squares. If the spiral center y_0 is a further unknown, the system may be written as the partitioned matrix

$$\begin{bmatrix} e^{rx_i} \cos(\omega x_i) & e^{rx_i} \sin(\omega x_i) & 1 & 0 \\ e^{rx_i} \sin(\omega x_i) & -e^{rx_i} \cos(\omega x_i) & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ \Re(y_0) \\ \Im(y_0) \end{bmatrix} = \begin{bmatrix} \Re(y_i) \\ \Im(y_i) \end{bmatrix}$$

Then amplitude and phase are given by

$$c = \sqrt{a^2 + b^2}$$

$$\phi = \arctan(b, a)$$

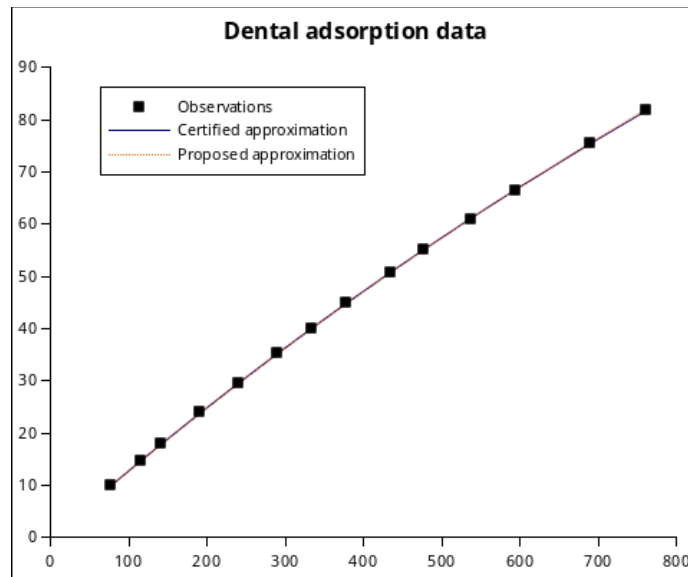
Example Problems

Implementation. The methods described in the prior section have been implemented in package LILS in FORTRAN with a C translation. This package is in the public domain and is available on the author's website at <http://13olive.net/code/lils1.zip> The following problems can be replicated with the sample programs included in that package.

Misra's dental adsorption data

These is a benchmark data set maintained by the National Institute of Standards and Technology (NIST) (USA). The measurements are of volume as a function of pressure, N=14, fit to a monomolecular (Mitscherlich / diminishing returns) curve. The optimal parameter values have been determined by NIST. Program dental fits the curve using the proposed integral method, obtaining parameter values that differ about 5% from the NIST certified results. A

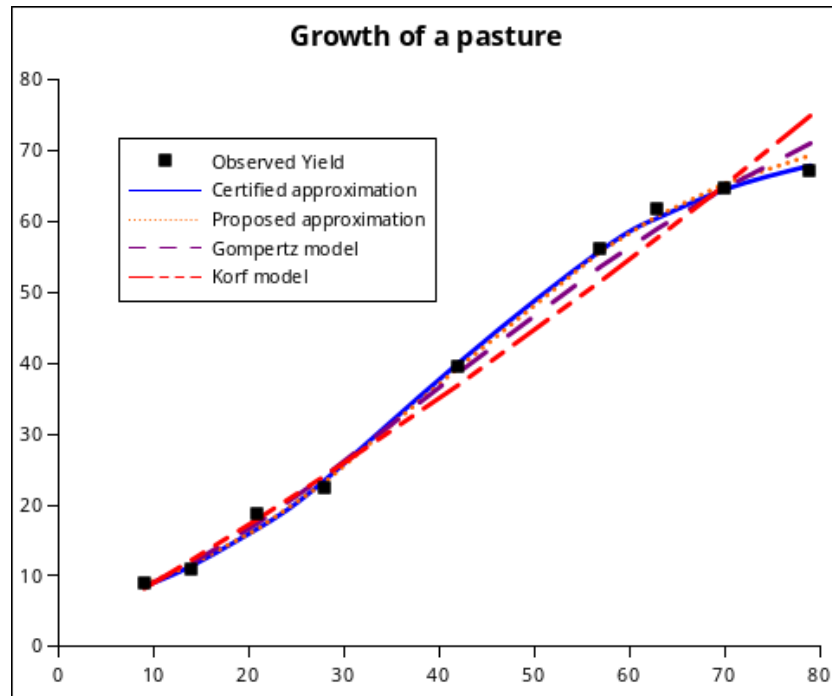
comparison of the graphs shows that the integral method performs well on this relatively easy problem.



Parameter	Certified value	Estimated value	Relative error
b1	238.942129	249.763741	4.53 %
b2	0.000550	0.000523	5.02 %

Ratkowsky's pasture yield data

This is another NIST benchmark data set, concerning the growth of a pasture as a function of time. The data was fit to a logistic curve and optimal parameters were determined by NIST. Program `pastur` fits this growth data to the logistic model according to the proposed integral method, and for sake of comparison, to the Gompertz and Korf models as well. The coefficient of determination is computed for each fit. There is close agreement between the certified and the integral approximations, and all these models give reasonable results.

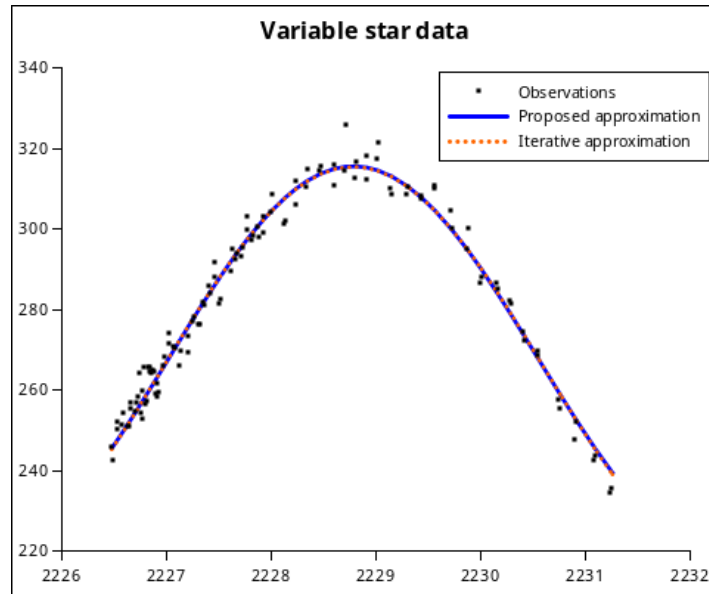


Model	Coefficient of determination
Logistic (NIST certified)	0.998267
Logistic (proposed method)	0.997483
Gompertz	0.992516
Korf	0.976315

Variable star data

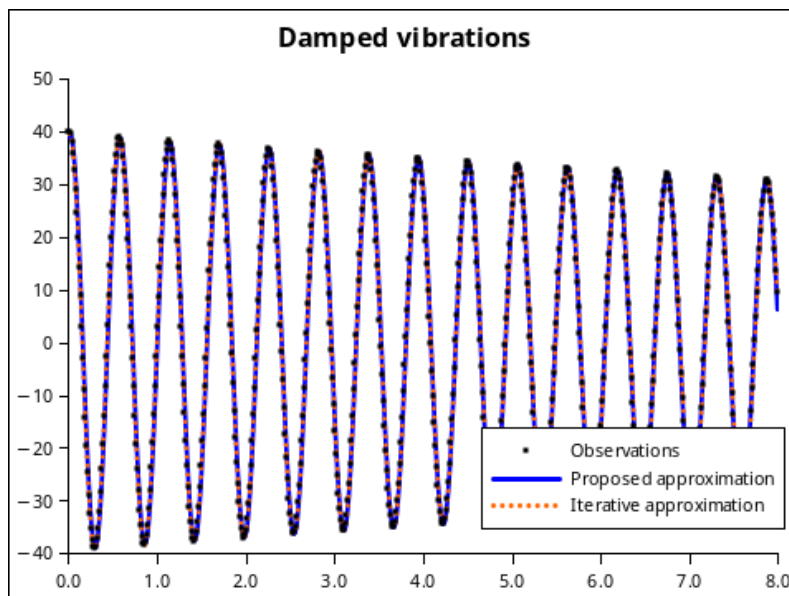
Data on the brightness of a variable star was published by the SuperWASP project. (Norton, et. al.) A variable star classified as a rotator is believed to vary in brightness across its surface and its changes in observed brightness are due to the star's rotation. This results in a case of simple harmonic variation in brightness as a function of time.

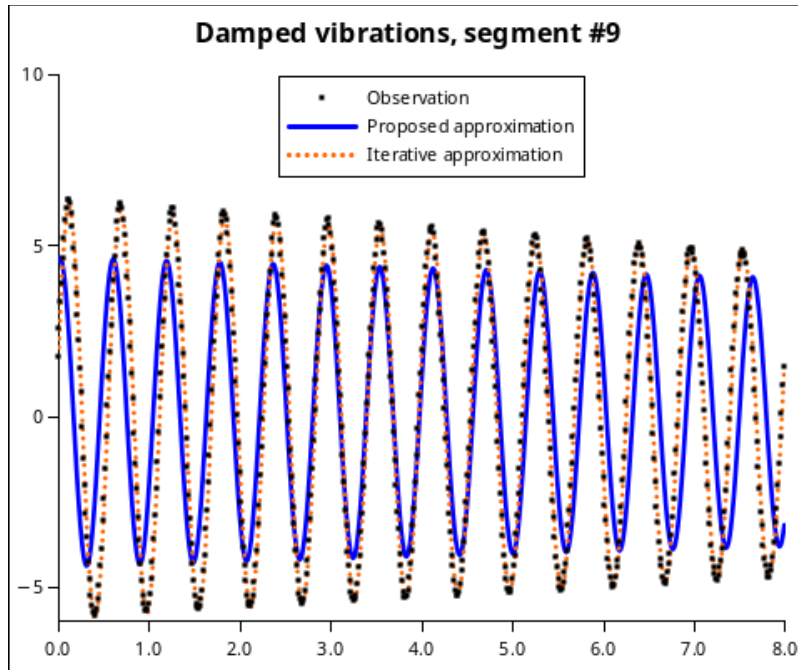
The integral method, unlike Fourier analysis, does not require equally spaced samples. It cannot tolerate long gaps in the data sequence, unfortunately, because this will corrupt the numerical integrals. The longest available sequence of observations taken over a single night was selected for study and the excerpt `var_star.csv` is included in the package. Program `star` determines the period of the variable star. The approximation determined by the proposed method was compared against an iteratively optimized fit computed by the spreadsheet `Gnumeric` (using algorithm `NLSOLVE`). The results show excellent agreement.



Gharib's damped vibrations data

Measurements of position versus time of a vibrating mass-damper system were published by Gharib (2021) in the course of his research in structural engineering. A portion of his data was excerpted as `damp.csv` and program `damp` fits the data according to the integral method, then assesses the fit according to the coefficient of determination. The program first attempts to fit the 80-second interval in one piece. This does not succeed. The program then breaks the record into ten 8-second segments and fits them separately. A satisfactory fit is found for most of the segments, however, the ninth has a C.O.D. of only 61%. The approximations over the first segment and the ninth segment are plotted below.



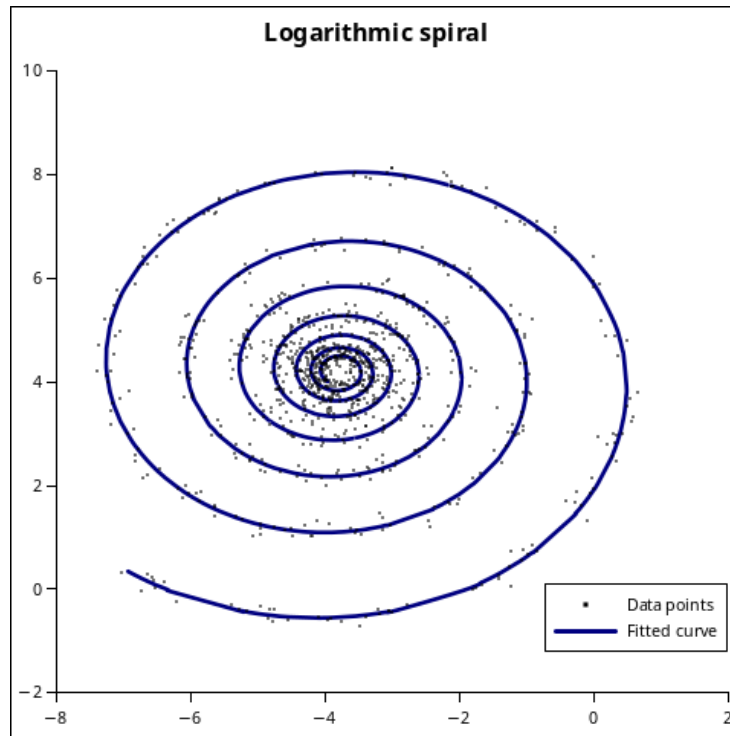


Segment	Coefficient of determination
1	0.998453
2	0.998402
3	0.997122
4	0.997135
5	0.998669
6	0.996201
7	0.994579
8	0.998284
9	0.618703
10	0.987325

What is particularly alarming about this result is that there is no obvious anomaly in the data to explain the poor results on this segment. So while the proposed integral method can be said to have very high computational efficiency, it cannot be claimed that it is entirely reliable.

Spiral problem

This is a toy problem. The file `spiral.csv` gives the coordinates of a spiral in the complex plane with additive Gaussian noise. Program `spiral` recovers an approximation to the original parameters. The coefficient of determination in the complex plane is 0.994.



Discussion and Conclusion

The integral method is noteworthy for its low computational burden – linear in its inputs. It serves well as an initial estimate for iterative procedures, but due to its sometimes fickle performance, cannot be recommended as a final answer. The method depends on the stability of integration for its effectiveness. In contrast, numerical differentiation is unstable, and does not form a basis for practical estimation methods. Cubic splines are an alternative to the trapezoid rule for numerical integration. They can yield higher precision, but are also more sensitive to noise. The trapezoid rule, the safer, low-order method, should be preferred in general.

Allowing for the well-understood operation of detrending, formulas for exponential growth rate and sinusoid frequency are simple and involve only two sums each, making these methods quite easy to implement.

The author is aware of little research on the topic, and it may be that mathematicians have only begun to scratch the surface of this powerful technique.

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[4b4ed1e691ff195be3ca73879a674234/Regressions-et-equations-integrales.pdf](https://scikit-guess.readthedocs.io/en/sine/_downloads/4b4ed1e691ff195be3ca73879a674234/Regressions-et-equations-integrales.pdf)

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Appendix

Algorithm RATEST

Given: data set (x_i, y_i) of length N

Assume: model equation of form $z = a + b e^{cx}$

Find: parameter c

Begin.

Compute numerical integral $s = \int y dx$ by the trapezoid rule.

$$s_1 = 0$$

$$s_{i+1} = s_i + \frac{1}{2}(y_{i+1} + y_i)(x_{i+1} - x_i)$$

Find the least-squares line $\hat{s}_{lin}$ to fit s as a linear function of x

$$\text{Detrend } \tilde{s} = s - \hat{s}_{lin}$$

$$\text{Compute } c = \frac{\sum_{i=1}^N \tilde{s}_i y_i}{\sum_{i=1}^N \tilde{s}_i^2}$$

Done.

Algorithm FREEST

Given: data set (x_i, y_i) of length N

Assume: model equation of form $z = a \cos(\omega x) + b \sin(\omega x) + c$

Find: parameter ω

Begin.

Compute numerical integral $s_1 = \int z dx$ by the trapezoid rule.

Compute numerical integral $s_2 = \int s_1 dx$ by the trapezoid rule.

Find least-squares parabola $(\hat{s}_2)_{prb}$ to fit s_2 as a quadratic function of x

$$\text{Detrend } \tilde{s}_2 = s_2 - (\hat{s}_2)_{prb}$$

$$\text{Compute } \omega = \sqrt{-\frac{\sum_{i=1}^N (\tilde{s}_2)_i y_i}{\sum_{i=1}^N (\tilde{s}_2)_i^2}}$$

Done.